$\underline{Ex}$: $\underline{A} = \begin{bmatrix} -11 & 0 & -4 \\ -1 & -9 & -1 \\ 1 & 0 & 1 \end{bmatrix}$, $|\underline{A} - \lambda \underline{I}| = 0 \Rightarrow \dots$

$\dots \Rightarrow \lambda = -9, -9, -9$

(algebraic multiplicity 3)

Solve $\left(\underline{A} - (-9)\underline{I}\right)\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} -2 & 0 & -4 \\ -1 & 0 & -1 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

augmented matrix

$$\left[\begin{array}{ccc|c} -2 & 0 & -4 & 0 \\ -1 & 0 & -1 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right]$$

$R_1 \overset{\text{swap}}{\longleftrightarrow} R_2$ $\left[\begin{array}{ccc|c} -1 & 0 & -1 & 0 \\ -2 & 0 & -4 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right]$

$\left.\begin{array}{l} R_1 \to (-1)R_1 \\ R_2 \to R_2 + 2R_3 \end{array}\right\}$ $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right]$

$R_2 \overset{\text{swap}}{\longleftrightarrow} R_3$ $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$R_2 \to R_2 - R_1$ $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

b free, a, c leading $\left\{ \begin{array}{l} a + c = 0 \Rightarrow a = 0 \\ c = 0 \end{array} \right.$

so solutions for $\left(\underline{A} - (-9)\underline{I}\right)\begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$ are

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix} = b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

only one basis vector, pick $b = 1$, $\underline{v_1} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Because $\lambda = -9$ has alg. mult. 3, but we found one lin. indep eigenvector (the $\underline{v_1}$ we just found),

$\dim(\mathcal{E}_{-9}) = 1 \Rightarrow$ geom.mult. 1, and so $\lambda = -9$ has defect $d = 3 - 1 = 2$. So we need to find 2 generalized e-vects $\underline{v}_2, \underline{v}_3$

Use (method 2) to make a "chain" of length 3, namely just want $\underline{v}_1, \underline{v}_2, \underline{v}_3 \neq 0$ such that

$$\begin{cases} (\underline{A} - \lambda I)^3 \underline{v}_3 = 0 \\ \underline{v}_2 := (\underline{A} - \lambda I)\underline{v}_3 & (\text{needs to be } \neq 0) \\ \underline{v}_1 := (\underline{A} - \lambda I)\underline{v}_2 & (\text{" " " "}) \end{cases}$$

Using $\underline{A}$ from example and $\lambda = -9$, we find that

$$(\underline{A} - (-9)I)^3 \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

so a, b, c free. Pick $a = 1$, $b = 0$, $c = 0$, $\underline{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Then

$$\underline{v}_2 = (\underline{A} - (-9)I)\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 & 0 & -4 \\ -1 & 0 & -1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ -1 \\ +1 \end{bmatrix} \quad (\neq 0)$$

$$\underline{v}_1 = (\underline{A} - (-9)I)\begin{bmatrix} -2 \\ -1 \\ +1 \end{bmatrix}$$

$$= \cdots = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

(which happens to be the same $\underline{v}_1$ we found earlier, but it isn't always like this)

What if we picked $a=0$, $b=1$, $c=0$ and made $\underline{v_3} := \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$?

Then
$$\underline{v_2} = (\underline{A} - (-9)I)\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} -2 & 0 & -4 \\ -1 & 0 & -1 \\ +1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

but we said $\underline{v_2}$ can't be 0. ($\underline{v_1}$ would also be $=0$ here, which also can't happen)

Hence we shouldn't use the choice $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

* for <u>this</u> problem.

$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ might work in other problems.

So we have $\underline{v_1} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\underline{v_2} = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$, $\underline{v_3} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Solutions
$$\mathbf{X}_1(t) := e^{\lambda t}\underline{v_1} = e^{-9t}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$
$$\mathbf{X}_1(t) := e^{\lambda t}(t\underline{v_1} + \underline{v_2})$$
$$= e^{-9t}\left(t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}\right)$$

compare/
notice
pattern

$$\mathbf{X}_3(t) = e^{\lambda t}\left(\frac{t^2}{2!}\underline{v_1} + t\,\underline{v_2} + \underline{v_3}\right)$$
$$= e^{-9t}\left(\frac{t^2}{2}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$$

General solution is
$$\mathbf{X}(t) = c_1\mathbf{X}_1(t) + c_2\mathbf{X}_2(t) + c_3\mathbf{X}_3(t).$$

If we needed $\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4 \neq 0$ (in a different problem)

such that $(\underline{A} - \lambda I)^4 \underline{v}_4 = 0$

and chain
$$\begin{cases} (\underline{A} - \lambda I)\underline{v}_4 = \underline{v}_3 \\ (\underline{A} - \lambda I)\underline{v}_3 = \underline{v}_2 \\ (\underline{A} - \lambda I)\underline{v}_2 = \underline{v}_1 \\ (\underline{A} - \lambda I)\underline{v}_1 = 0 \end{cases},$$

then we'd have everything like before, but also add-in

$$\underline{X}_4(t) = e^{\lambda t}\left(\frac{t^3}{3!}\underline{v}_1 + \frac{t^2}{2!}\underline{v}_2 + t\underline{v}_3 + \underline{v}_4\right)$$

However, these are lengthy and don't show up in problems too much.